

SUBSETS OF FULL MEASURE IN A GENERIC SUBMANIFOLD IN \mathbb{C}^n ARE NON-PLURITHIN

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Abstract : In this paper we prove that if $I \subset M$ is a subset of measure 0 in a C^2 –smooth generic submanifold $M \subset \mathbb{C}^n$, then $M \setminus I$ is non-plurithin at each point of M in \mathbb{C}^n . This result improves a previous result of A. Edigarian and J. Wiegerinck who considered the case where I is pluripolar set contained in a C^1 –smooth generic submanifold $M \subset \mathbb{C}^n$ ([EW10]). The proof of our result is essentially different.

Key words : Generic manifold, attached analytic discs, plurisubharmonic function, pluripolar set, plurithin set.

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1. INTRODUCTION

Real p –planes $\Pi \subset \mathbb{C}^n$, $\dim_R \Pi = p$, $p \in \mathbb{N}$, which are not contained in any proper complex subspace of \mathbb{C}^n are important in complex analysis and pluripotential theory. The \mathbb{C} –hull of such plane Π is equal to all \mathbb{C}^n i.e. $\Pi + J\Pi = \mathbb{C}^n$ (J is the standard complex structure on \mathbb{C}^n) and any non empty open subset of Π is non pluripolar in \mathbb{C}^n . Such planes are called generic (real) subspaces of \mathbb{C}^n . Correspondingly, a real smooth submanifold $M \subset \mathbb{C}^n$ is said to be generic if for each $z \in M$, its real tangent space $T_z M$ is a generic subspace of \mathbb{C}^n i.e. $T_z M + JT_z M = \mathbb{C}^n$. Such submanifold has real dimension $m \geq n$.

The case of minimal dimension $\dim M = n$ is the most relevant. In this case for each $z \in M$, the tangent space $T_z M$ does not contain any complex line i.e. $T_z M \cap JT_z M = \{0\}$ and M is said to be *maximal totally real*.

One of the main notions in classical potential theory related to the Dirichlet problem is the notion of thinness. The corresponding notion in pluripotential theory can be defined as follows.

Definition. A subset $E \subset \mathbb{C}^n$ is plurithin at $z^0 \in \overline{E}$ if there exists a plurisubharmonic function u in a neighbourhood of z^0 in \mathbb{C}^n such that

$$u(z^0) > \varlimsup_{z \rightarrow z^0, z \in M \setminus I} u(z).$$

The main result of this paper is the following.

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Theorem. *Let $M \subset \mathbb{C}^n$ be a C^2 generic submanifold and let $I \subset M$ be an arbitrary subset of measure zero in M . Then $M \setminus I$ is non-plurithin at any point of M .*

A. Edigarian and J. Wiergerinck proved that if $M \subset \mathbb{C}^n$ be a C^1 -smooth generic submanifold and $P \subset \mathbb{C}^n$ is a pluripolar set, then $M \setminus P$ is non-plurithin at any point of M (see [EW10]).

By a result of the first author ([Sa76]) for C^3 -smooth manifolds, extended to the C^2 -smooth case by B. Coupet ([C92]), any pluripolar set $P \subset M$ is of measure zero in M . Therefore for C^2 -smooth submanifolds, our result improves the result of A. Edigarian and J. Wiergerinck.

A powerfull method for studying the *pluripotential* properties on generic submanifolds (e.g. *pluripolarity, pluriregularity or plurithinness*) is the well known method of attaching analytic disks (see [B65], [Pin74], [KhCh76], [Sa76], [C92], [EW10]). Here we will use the same method but we need to establish more precise properties of the smooth family of analytic disks attached to M along a half of the unit circle.

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2. BISHOP'S CONSTRUCTION

Let us recall Bishop's method for constructing analytic discs attached to real submanifolds.

Let $M \subset \mathbb{C}^n$ be a totally real C^k -submanifold ($k \geq 1$) of dimension n given locally by the following equation

$$(2.1) \quad M := \{z = x + iy \in B \times \mathbb{R}^n : y = h(x)\},$$

where $B \subset \mathbb{R}^n$ is an euclidean ball of center 0 and $h : B \rightarrow \mathbb{R}^n$ is a C^2 function such that

$$h(0) = 0 \quad \text{and} \quad Dh(0) = 0.$$

Let $v : \mathbb{T} \rightarrow \mathbb{R}^+$ a C^∞ function on the unit circle \mathbb{T} such that

$$(\star) \quad v|_\gamma = 0 \quad \text{and} \quad v|_{\mathbb{T} \setminus \gamma} > 0,$$

where $\gamma := \{e^{i\theta} : \theta \in [0, \pi]\}$.

Assume that there exist a continuous mapping $X : \mathbb{T} \rightarrow \mathbb{R}^n$ solution of the following Bishop equation

$$(2.2) \quad X(\tau) = c - \Im(h \circ X + tv)(\tau), \quad \tau \in \mathbb{T},$$

where $(c, t) \in Q = Q_c \times Q_t \subset \mathbb{R}^n \times \mathbb{R}^n$ is a fixed parameter and \Im is the harmonic conjugate operator defined by the Schwarz integral formula

$$(2.3) \quad \Im(X)(\varsigma) = \frac{1}{2\pi} \int_T X(\tau) \operatorname{Im} \frac{e^{i\tau} + \varsigma}{e^{i\tau} - \varsigma} d\tau, \quad \varsigma = re^{i\theta},$$

normalized by the condition

$$\Im X(0) = 0.$$

We will consider the unique harmonic extension $X(\varsigma)$ of the mapping $X(\tau)$ to the unit disk \mathbb{U} . Then the following holomorphic mapping

$$\begin{aligned} \Phi(c, t, \varsigma) &:= X(c, t, \varsigma) + i[h^*(c, t, \varsigma) + tv(\varsigma)] = \\ &= c + i\{h^*(c, t, \varsigma) + tv(\varsigma) + i\Im[h^*(c, t, \varsigma) + tv(\varsigma)]\} \end{aligned}$$

provides a family of analytic disks $\Phi(c, t, \cdot) : \overline{\mathbb{U}} \rightarrow \mathbb{C}^n$ attached to M along the arc $\gamma = \{e^{i\theta} : \theta \in [0, \pi]\}$ in the following sense

$$\forall (c, t) \in Q, \quad \forall \tau \in \gamma, \quad \Phi(c, t, \tau) \in M.$$

Here $X(c, t, \cdot)$, $h^*(c, t, \cdot)$ and v denote the harmonic extensions of $X(c, t, \tau)$, $h \circ X(c, t, \tau)$ and $v(\tau)$ from the unit circle \mathbb{T} to the unit disk \mathbb{U} , respectively.

We need a smooth family $\Phi(c, t, \cdot)$ of analytic discs. This is provided by the following result of B.Coupet ([C10]) extending a construction done previously by the first author.

Theorem ([C92]). *Let $h \in C^k(\mathbb{B})$, $k \in \mathbb{N}^*$. Then there exist a neighborhood $Q = Q_c \times Q_t \ni 0$ such, that the Bishop equation (2.2) has a unique solution $u \in C^{k-1}(Q \times \mathbb{T})$.*

Moreover, the harmonic extensions of u , $h \circ u$ to the unit disk \mathbb{U} , their conjugates $\Im u$, $\Im h \circ u$ belong $C^k(Q \times \mathbb{U}) \cap C^{k-1}(Q \times \overline{\mathbb{U}})$ and

$$\|\Im u\| \leq A \|u\|, \quad \|\Im h\| \leq A \|h\|,$$

where A is a constant.

Therefore, when the submanifold M is C^1 -smooth, the solution $X(\tau, c, t)$ is continuous in $Q \times \overline{\mathbb{U}}$ and for twice smooth submanifold we obtain a C^1 -smooth family of disks, attached to M .

In the case when $I \subset M$ is pluripolar, Edigarian and Wiegerinck [EW] needed only a continuous family of disks and gave a beautiful proof for the non-thinness of $M \setminus I$.

Recall that every pluripolar set $I \subset M$ on the twice smooth generic manifold M has zero-measure (see [Sa76], [C92]), but the converse is far from being true. Indeed, there are many subsets of $\mathbb{R}^n \subset \mathbb{C}^n$ with zero-measure which are not pluripolar in \mathbb{C}^n . So our theorem is a non trivial improvement of Edigarian and Wiegerinck theorem in the case of C^2 -manifolds. We need to assume C^2 -smoothness of M because we need smoothness of the family of discs constructed in ([C92]).

Observe that the family of discs $\Phi(c, t, \zeta)$ obtained above

$$(2.4) \quad \Phi(c, t, \zeta) = X(c, t, \zeta) + i(h^*(c, t, \zeta) + tv(\zeta)), \quad (c, t) \in Q, \zeta \in \bar{\mathbb{U}},$$

satisfies the following conditions:

$$(2.5) \quad X(c, t, \tau) = c - \Im(h \circ X(c, t, \tau) + tv(\tau)), \quad (c, t) \in Q, \tau \in \partial\mathbb{U}.$$

$$(2.6) \quad h^*(c, t, \tau) = h \circ X(c, t, \tau), \quad (c, t) \in Q, \tau \in \partial\mathbb{U}$$

$$(2.7) \quad \begin{aligned} X(c, 0, \zeta) &\equiv c, h^*(c, 0, \zeta) \equiv h(c) \text{ so that} \\ \Phi(c, 0, \zeta) &\equiv c + ih(c) \in M, c \in Q_c \end{aligned}$$

$$(2.8) \quad X(c, t, 0) = \frac{1}{2\pi} \int_T X(c, t, \tau) d\tau \equiv c, \quad (c, t) \in Q.$$

3. PROOF OF THE THEOREM

1. First case. We consider the case when $\dim M = n$. Fix a point p , say $p = 0 \in M$. We want to prove that $M \setminus I$ is non-plurithin at 0. Assume the contrary, that there exist a $V(z) \in PSH(G)$ such that $V(0) > 1/2$, $V|_{M \setminus I \cup \{0\}} = 0$ and $0 \leq V(z) \leq 1$, for $z \in G$, where G a neighbourhood of $0 \in \mathbb{C}^n$.

The proof of the theorem goes in several steps.

1⁰. First, for convenience, we introduce the following terminology. We say that the disc $\Phi_{c,t} := \Phi(c, t, \cdot)$, defined by (2.4) is *good* if

$$H_1(\gamma_I) = 0, \text{ where } \gamma_I := \{\tau \in \gamma : \Phi(c, t, \tau) \in I\}.$$

Assume that for some fixed value of the parameter $(c, t) \in Q$, the corresponding disc $\Phi_{c,t}$ is "good". Then the function defined on \mathbb{U} by the formula $u(\zeta) := V \circ \Phi(c, t, \zeta)$ satisfies the following properties (we note that $\Phi(c, t, \cdot) : \bar{\mathbb{U}} \rightarrow G$ for a small enough Q):

$$u \in SH(\mathbb{U}) \cap C(\bar{\mathbb{U}} \cup \{\gamma \setminus \gamma_I\}), \quad u|_{\gamma \setminus \gamma_I} = 0.$$

Let $\omega(\zeta, \gamma \setminus \gamma_I, \mathbb{U})$ be the harmonic measure of the set $\gamma \setminus \gamma_I$ with respect to \mathbb{U} at the point ζ . Then $\omega(\zeta, \gamma \setminus \gamma_I, \mathbb{U})$ is the generalized solution of the Dirichlet problem in \mathbb{U} with boundary data equal to $-\chi_{\gamma \setminus \gamma_I}$ on $\partial\mathbb{U}$, where $\chi_{\gamma \setminus \gamma_I}$ is the characteristic function of the Borel set $\gamma \setminus \gamma_I$. This means that $\omega(\zeta, \gamma \setminus \gamma_I, \mathbb{U})$ is a harmonic function negative on \mathbb{U} and equal to $-\chi$ quasi-everywhere on $\partial\mathbb{U}$, in particular $\omega(\zeta, \gamma \setminus \gamma_I, \mathbb{U}) = -1$ on $\gamma \setminus \gamma_I$. Since $H_1(\gamma_I) = 0$, it follows from Poisson integral formula that $\omega(\zeta, \gamma \setminus \gamma_I, \mathbb{U}) \equiv \omega(\zeta, \gamma, \mathbb{U})$, since they coincide almost everywhere on $\partial\mathbb{U}$.

We put

$$\Omega = \{\zeta \in \bar{\mathbb{U}} : 1 + \omega(\zeta, \gamma, \mathbb{U}) < 1/4\}.$$

Then $\Omega \supset \gamma$ and $\Omega = \tilde{\Omega} \cap \overline{\mathbb{U}}$ for some open set $\tilde{\Omega} : \gamma \subset \tilde{\Omega}$. We note that Ω depended only of γ . By the so called two-constant theorem, it follows that for any $\varsigma \in \mathbb{U}$

$$(3.1) \quad 0 \leq u(\zeta) \leq 1 + \omega(\zeta, \gamma, \mathbb{U}).$$

This implies that $u \equiv 0$ on γ and then

$$(3.2) \quad V(z) < 1/4,$$

provided that the point z lies in the image $\Phi_{c,t}(\Omega)$ of the good disk $\Phi_{c,t}$.

2⁰. The next step is to prove that for arbitrary fixed $\zeta^0 \in \Omega \cap \mathbb{U}$ the images $\Phi(c, t, \zeta^0)$, as (c, t) vary in Q , fill up an open set $W(\zeta^0) \ni 0$ in $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$.

Consider the map

$$S : (c, t) \rightarrow \Phi(c, t, \zeta^0) = X(c, t, \zeta^0) + i[h^*(c, t, \varsigma^0) + tv(\varsigma^0)].$$

The jacobian of S is

$$J(c, t) = \begin{vmatrix} D_{c_j} X & \dots & D_{t_j} X \\ \dots & \dots & \dots \\ D_{c_j} h^* & \dots & v(\zeta^0) \mathbb{I} + D_{t_j} h^* \end{vmatrix}$$

where \mathbb{I} is the identity matrix in \mathbb{R}^n . We see that

$$J(c, t) = v^n(\varsigma^0) \frac{\partial X_1}{\partial c_1} \dots \frac{\partial X_n}{\partial c_n} + O \left(\sum_{i \neq j} \|D_{c_j} X_i\|, \sum_j [\|D_{c_j} h^*\| + \|D_{t_j} h^*\|] \right)$$

Here and below $\|\cdot\|$ denote the *sup* norms.

If $t = 0$ then by (2.7) $X(c, 0, \zeta^0) \equiv c$ and $h^*(c, 0, \zeta^0) \equiv h(c)$. Hence by smoothness of Φ and the condition (\star) , it follows that

$$J \neq 0$$

for small enough Q_c and Q_t (remember that $h(0) = Dh(0) = 0$). It follows that the transformation S is open and then the following set

$$W(\zeta^0) = S(Q) = \{\Phi(c, t, \zeta^0) : (c, t) \in (Q)\}$$

is open in $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ and contains 0.

Shrinking, if necessary, Q we can assume, that $W(\zeta) \subset G$, $\forall \zeta \in \Omega \cap \mathbb{U}$ and put $W = \bigcup \{W(\zeta) : \zeta \in \Omega \cap \mathbb{U}\}$.

3⁰. Set

$$W' = \bigcup_{c,t} \{W \cap \Phi_{c,t}(\Omega) : \Phi_{c,t} \text{ is a good disc}\}.$$

Then (3.2) implies that $V(z) < 1/4$ for $z \in W'$.

Assume for the moment that $H_{2n}(W \setminus W') = 0$ (see next steps $3^0 - 5^0$). Then it follows that from the submean value property of subharmonic functions in \mathbb{R}^{2n} that $V(z) < 1/4$ everywhere on W° , i.e. for any $z^0 \in W^\circ$,

$$\limsup_{z \rightarrow z^0, z \in W'} V(z) = V(z^0)$$

In particular, $V(0) < 1/4$. This contradiction will prove the theorem in the case when $\dim M = n$.

The question remains to know "how much" good disks we have? In fact we have enough good discs to prove that $H_{2n}(W \setminus W') = 0$. Indeed, we calculate $D_\tau X(c, t, \tau)$, $(c, t) \in Q$, $\tau \in T$

$$(3.3) \quad D_\tau X(c, t, \tau) = -D_\tau \Im h \circ X(c, t, \tau) - t D_\tau \Im v(\tau)$$

Since, $D_\tau \Im v(\tau) = D_{\vec{n}} v(\tau)$, where $D_{\vec{n}}$ is the normal derivative \vec{n} , then

$$D_\tau X(c, t, \tau) + t D_{\vec{n}} v(\tau) = \Im D_\tau h \circ X(c, t, \tau).$$

For k -coordinate of vector $X(\tau) = X(c, t, \tau)$ we have

$$(3.4) \quad \begin{aligned} \|D_\tau X_k(c, t, \tau) + t_k D_{\vec{n}} v(\tau)\| &= \|\Im D_\tau h_k \circ X(c, t, \tau)\| \leq \\ &\leq \text{const} \|D_\tau h_k \circ X(c, t, \tau)\| \leq O(\varepsilon) \|D_\tau X(c, t, \tau)\|, \end{aligned}$$

where $\varepsilon = \{\|c\| + \|t\| : c \in Q_c, t \in Q_t\}$ and $\|(a_1, \dots, a_n)\| = \max\{|a_1|, \dots, |a_n|\}$.

Therefore,

$$(3.5) \quad \begin{aligned} |t_k D_{\vec{n}} v(\tau)| - O(\varepsilon) \|t\| &\leq |D_\tau X_k(c, t, \tau)| \leq \\ &\leq |t_k D_{\vec{n}} v(\tau)| + O(\varepsilon) \|t\|, \quad 1 \leq k \leq n, \quad \tau \in T \end{aligned}$$

The second part of (3.5) implies

$$(3.6) \quad \|D_\tau X(c, t, \tau)\| \leq C \|t\|, \quad (c, t, \tau) \in Q \times T,$$

where $C > 0$ is a constant.

If we denote by $b = \inf_{\gamma^0} |D_{\vec{n}} v(\tau)| > 0$, where $\gamma^0 \subset \subset \gamma$, then decreasing $\varepsilon > 0$ in (3.5) we can arrange so that $O(\varepsilon) < \frac{b}{2}$ and the first part of (3.5) implies

$$(3.7) \quad |D_\tau X_k(c, t, \tau)| \geq |t_k| b - \|t\| b/2,$$

for $\tau \in \gamma^0, 1 \leq k \leq n$.

4⁰. Fix $(x^0, y^0) \in W \setminus M$. Then by p. 2⁰ there exist $(c^0, t^0, \zeta^0) \in Q \times (\Omega \cap \mathbb{U})$ such that $\Phi(c^0, t^0, \zeta^0) = (x^0, y^0)$. Let $\|t^0\| = |t_k^0|$, for simplicity we assume $k = n$, and let $'c = (c_1, \dots, c_{n-1})$, $'t = (t_1, \dots, t_{n-1})$.

We consider the transformation

$$(3.8) \quad S : ('c, 't, \varsigma) \rightarrow \Phi('c, c_n^0, 't, t_n^0, \varsigma) : 'Q \times \overline{\mathbb{U}} \rightarrow \mathbb{C}^n,$$

where

$$'Q := Q \cap \{c_n = c_n^0, t_n = t_n^0\} \subset \mathbb{R}^{2n-2}.$$

Then $S(c^0, t^0, \varsigma^0) = (x^0, y^0)$ and its jacobian is equal to

$$J(c, t, \varsigma) = \begin{vmatrix} \frac{\partial x_1}{\partial c_1} & \cdots & \frac{\partial x_{n-1}}{\partial c_1} & \left| \right. & \frac{\partial y_1}{\partial c_1} & \cdots & \frac{\partial y_{n-1}}{\partial c_1} & \left| \right. & \frac{\partial x_n}{\partial c_1} & \frac{\partial y_n}{\partial c_1} \\ \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots \\ \frac{\partial x_1}{\partial c_{n-1}} & \cdots & \frac{\partial x_{n-1}}{\partial c_{n-1}} & \left| \right. & \frac{\partial y_1}{\partial c_{n-1}} & \cdots & \frac{\partial y_{n-1}}{\partial c_{n-1}} & \left| \right. & \frac{\partial x_n}{\partial c_{n-1}} & \frac{\partial y_n}{\partial c_{n-1}} \\ \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots \\ \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_{n-1}}{\partial t_1} & \left| \right. & \frac{\partial y_1}{\partial t_1} & \cdots & \frac{\partial y_{n-1}}{\partial t_1} & \left| \right. & \frac{\partial x_n}{\partial t_1} & \frac{\partial y_n}{\partial t_1} \\ \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_{n-1}}{\partial t_{n-1}} & \left| \right. & \frac{\partial y_1}{\partial t_{n-1}} & \cdots & \frac{\partial y_{n-1}}{\partial t_{n-1}} & \left| \right. & \frac{\partial x_n}{\partial t_{n-1}} & \frac{\partial y_n}{\partial t_{n-1}} \\ \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots & \cdots & \left| \right. & \cdots & \cdots \\ \frac{\partial x_1}{\partial \varsigma'} & \cdots & \frac{\partial x_{n-1}}{\partial \varsigma'} & \left| \right. & \frac{\partial y_1}{\partial \varsigma'} & \cdots & \frac{\partial y_{n-1}}{\partial \varsigma'} & \left| \right. & \frac{\partial x_n}{\partial \varsigma'} & \frac{\partial y_n}{\partial \varsigma'} \\ \frac{\partial x_1}{\partial \varsigma''} & \cdots & \frac{\partial x_{n-1}}{\partial \varsigma''} & \left| \right. & \frac{\partial y_1}{\partial \varsigma''} & \cdots & \frac{\partial y_{n-1}}{\partial \varsigma''} & \left| \right. & \frac{\partial x_n}{\partial \varsigma''} & \frac{\partial y_n}{\partial \varsigma''} \end{vmatrix},$$

Here $\varsigma = \varsigma' + i\varsigma''$ and

$$x_k(c, t, \varsigma) = X_k(c, c_n^0, t, t_n^0, \varsigma), \quad k = 1, \dots, n,$$

$$y_k(c, t, \varsigma) = h_k^* \circ X(c, c_n^0, t, t_n^0, \varsigma) + t_k v(\varsigma), \quad k = 1, \dots, n-1,$$

$$y_n(c, t, \varsigma) = h_n^* \circ X(c, c_n^0, t, t_n^0, \varsigma) + t_n^0 v(\varsigma).$$

The determinant J , is composed by 9 block matrices D_{ij} , $i, j = 1, 2, 3$.

We will show that $J(c^0, t^0, \varsigma^0) \neq 0$, which will imply that the operator S is local diffeomorphism in a neighbourhood of the point (c^0, t^0, ς^0)

By (2.7) $X(c, 0, \varsigma) \equiv c$, $h^*(c, 0, \varsigma) \equiv h(c)$. Therefore,

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}_{(c, 0, \varsigma)} = D_{11} \cdot D_{22} = v^{n-1}(\varsigma)$$

and

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}_{(c, t, \varsigma)} = v^{n-1}(\varsigma) + O(\varepsilon).$$

Note also that if $\varsigma = \varsigma' + i\varsigma''$ then

$$(3.9) \quad D_{33} = \begin{vmatrix} \frac{\partial x_n}{\partial \varsigma'} & \frac{\partial y_n}{\partial \varsigma'} \\ \frac{\partial x_n}{\partial \varsigma''} & \frac{\partial y_n}{\partial \varsigma''} \end{vmatrix} = \left| \frac{d}{d\varsigma}(x_n + iy_n) \right|^2.$$

Now consider the right hand side near the arc γ^0 . It is clear that for every $s > 0$, there is an open set $\Omega' \supset \gamma^0$ such that

$$\left| \frac{d}{d\varsigma}(x_n + iy_n)(c, t, \varsigma) \right|^2 \geq |D_\tau x_n(c, t, \tau)|^2 - s, \quad \forall \varsigma \in \mathbb{U} \cap \Omega', \tau \in \gamma^0, (c, t) \in Q.$$

By (3.6), (3.7) and (3.9), it follows that

$$\begin{aligned} |J(c, t, \varsigma)| &= |D_{11}| \cdot |D_{22}| \cdot \left| \frac{d}{d\varsigma}(x_n + iy_n)(c, t, \varsigma) \right|^2 + O(\varepsilon) \\ &\geq [v^{n-1}(\varsigma) + O(\varepsilon)] \cdot [|t_n b/2|^2 - s] + O(\varepsilon), \end{aligned}$$

for all $('c, 't, \zeta) \in 'Q \times [\mathbb{U} \cap \Omega']$.

We can take $\Omega \cap \Omega'$ instead of Ω and observe that all functions $O(\varepsilon)$ do not depend on ξ . Therefore if we take ε, s small enough, then $|J('c^0, 't^0, \xi^0)| > 0$ and, in particular, the operator

$$S('c, 't, \zeta) : ' \hat{Q} \times \{|\zeta - \zeta_0| < \sigma'\} \rightarrow \mathbb{U}(x^0, y^0)$$

is an homeomorphism, where $\sigma' > 0$ and $' \hat{Q} = ' \hat{Q}_c \times ' \hat{Q}_t \subset 'Q$ is a neighbourhood of $('c^0, 't^0)$ and $U(x^0, y^0)$ is a neighbourhood of (x^0, y^0) .

5⁰. We show that there exist a neighborhood $\check{Q}_t \subset ' \hat{Q}_t$ such that for every fixed $'t \in \check{Q}_t$ the mapping

$$S('c, 't, \tau) = \Phi('c, c_n^0, 't, t_n^0, \tau) : 'Q_c \times \gamma^0 \rightarrow M$$

is local homeomorphism. We put

$$A_k = \begin{pmatrix} \frac{\partial x_1}{\partial c_1} & \cdots & \frac{\partial x_n}{\partial c_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial c_{n-1}} & \cdots & \frac{\partial x_n}{\partial c_{n-1}} \end{pmatrix}^{\sqrt{k}},$$

where $(\dots)^{\sqrt{k}}$ means that the k^{th} column is omitted. Then $A_1 = \dots = A_{k-1} = 0$, $A_n = 1$ if $t = 0$. We consider the jakobian-minor

$$\begin{aligned} J('c, 't, \tau) &= \text{mod} \begin{vmatrix} \frac{\partial x_1}{\partial c_1} & \cdots & \frac{\partial x_n}{\partial c_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial c_{n-1}} & \cdots & \frac{\partial x_n}{\partial c_{n-1}} \\ \frac{\partial x_1}{\partial \tau} & \cdots & \frac{\partial x_n}{\partial \tau} \end{vmatrix} \geq \left| \frac{\partial x_n}{\partial \tau} \right| \cdot |A_n| - \left(\left| \frac{\partial x_1}{\partial \tau} \right| |A_1| + \dots + \left| \frac{\partial x_{n-1}}{\partial \tau} \right| |A_{n-1}| \right) \\ &= |A_n| \left\{ \left| \frac{\partial x_n}{\partial \tau} \right| - \left(\left| \frac{\partial x_1}{\partial \tau} \right| \frac{|A_1|}{|A_n|} + \dots + \left| \frac{\partial x_{n-1}}{\partial \tau} \right| \frac{|A_{n-1}|}{|A_n|} \right) \right\}. \end{aligned}$$

By (3.6) and (3.7) we have

$$\|D_\tau X(c, t, \tau)\| \leq C \|t\|, \quad (c, t, \tau) \in Q \times T, \quad C - \text{constant},$$

$$|D_\tau X_n(c, t^0, \tau)| \geq |t_n^0|b - \|t^0\|b/2 = \left(b - \frac{b}{2}\right) |t_n^0| = \frac{b\|t^0\|}{2}, \quad (c, t, \tau) \in Q \times T.$$

Therefore, $J('c, 't^0, \tau) \neq 0$ if t^0 small. The same is true for a neighborhood $\check{Q}_t \subset ' \hat{Q}_t$.

Since the set $I \subset M$ has zero-measure, then for almost every $'c \in 'Q_c$ the disks $\Phi('c, 't, \tau)$, with fixed $'t \in \check{Q}_t$, intersects I on zero-length set, i.e. the disks $\Phi('c, 't, \zeta)$ are "good" for almost every $'c \in 'Q_c$. Therefore, the $(W \setminus W') \cap \{\Phi('c, 't, \zeta) \in \mathbb{R}^{2n} : 'c \in 'Q_1, \zeta \in B(\zeta^0, r)\}$, $r > 0$, has zero $(n+1)$ -measure. It follows that in a neighbourhood of $(x^0, y^0) \in W$ the set $W \setminus W'$ has zero $2n$ -measure, which completes the proof that $H_{2n}(W \setminus W') = 0$, since (x_0, y_0) is arbitrarily fixed.

2. General case. Let M be an arbitrary generic manifold of dimension $m > n$ and let $I \subset M$ be a subset of measure zero in M .

Fix a point, say $z^0 = 0 \in M$. Changing holomorphic coordinates in \mathbb{C}^n , we can assume that the tangent space T_0M , which by definition does not contain any complex hyperplane, can be written as

$$T_0M = \{z = x + iy \in \mathbb{C}^n : y_1 = \cdots = y_{2n-m} = 0\}.$$

Hence for a small neighbourhood $G = G_1 \times G_2$ of the origin with

$$G_1 = \{(x, y'') = (x, y_{2n-m+1}, \dots, y_n) \in \mathbb{R}^n \times \mathbb{R}^{m-n} : |x| \leq \delta, |y''| < \delta\},$$

$$G_2 = \{y' = (y_1, \dots, y_{2n-m}) \in \mathbb{R}^{2n-m} : |y'| < \delta\},$$

we can represent M as a graph

$$M \cap G = \{z \in G : y' = h(x, y'')\},$$

where h is C^2 smooth mapping from G_1 into \mathbb{R}^{2n-m} .

Observe that for each $(m-n) \times n$ matrix B the intersection $M \cap \Pi_B$ of M with the plane $\Pi_B := \{z \in \mathbb{C}^n : y'' = x \cdot B\}$ is an n -dimensional generic manifold for any small enough B . Since $H_m(I) = 0$ then there exist B such that $I \cap \Pi_B$ has zero-measure. Hence there exists at least one generic submanifold M' of dimension n such that $0 \in M' \subset M$ and $H_n(I \cap M') = 0$.

By the first case the set $M' \setminus I$ is not thin at 0. It follows, that $M \setminus I$ also is not thin at 0. \triangleright

Open problem : Let $M \subset \mathbb{C}^n$ be a smooth generic submanifold of dimension m and $E \subset M$ a given Borel subset such that $0 \in E$ and the following condition holds

$$\lim_{r \rightarrow 0} \frac{H_m(E \cap \mathbb{B}_r)}{r^m} = 0,$$

where H_m is the Hausdorff measure of dimension m and $\mathbb{B}_r \subset \mathbb{C}^n$ is the euclidean ball of radius r . Then is it true that the set $M \setminus E$ non-plurithin at 0?

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